Anticipating Student Responses to Improve Problem Solving

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I HAVE BEEN A MATHEMATICS TEACHER AND EDUCATOR THROUGHOUT ALL FOUR publications of the *Standards* documents (NCTM 1991, 1995, 1989, and 2000). Over the years, while concentrating on improving various aspects of my teaching, specifically, improving my students' ability to problem solve, I have been perplexed to see students pick numbers out of a problem and perform an operation with no regard for the context. To address this issue by teaching problem-solving lessons made me realize that I did not know the difference between students solving a problem and actual problem solving. A lesson beginning with a problem or task does not make it a problem-solving lesson, especially when students would inevitably solve it the way I had intended. Instead of problem solving, my students were trying to figure out what I was thinking. To prevent the temptation of leading students in this way of thinking required careful planning of problem-solving lessons.

I was influenced by the research of Stigler, Fernandez, and Yoshida (1996), who contrasted American and Japanese teachers' lesson planning. They found that American les-

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son plans are typically presented in an outline form; a topic is presented and key points are bulleted. The key points primarily address definitions and formulas. Japanese lesson plans are much more detailed. They provide a context for the current lesson within previous and subsequent lessons as well as probing questions, possible student responses to the questions, and follow-up suggestions that are based on student responses. Stigler, Fernandez, and Yoshida found that teaching could be improved by using a detailed lesson plan format. I adapted a similar format for my problem-solving lessons, with components intended to provide a framework for my own metacognitive thinking. The framework is not scripted but includes sequenced learning activities; mathematical content questions; suggestions for possible student responses, correct and incorrect; misconceptions that students may have; and potential guidance that can be provided, based on student understanding. The lesson plans revolve around one problem. The specific questions presented are to help students understand the problem and get started solving it. Content-pedagogy questions for stimulating teacher-student and student-student interactions are also included.

Anticipating responses to the questions prevents me from influencing my students toward a particular path. Trying to anticipate what a student might think is quite a challenge. To aid the planning, I presented the same problem to several different mathematics methods classes comprising preservice elementary school and middle school education majors. I knew that I had to remain open and flexible to their thinking and reasoning so as to plan for students' responses. By continually revising my lesson plan based on new or different points and views from the students and reflecting each time I presented the problem forced me to consciously think about the problem-solving experience. This process allowed me to think ahead about how to facilitate students' understanding of the problems.

To help my preservice teachers write a lesson plan using this format, they were asked to choose a problem for their upper-elementary and middle school practicum students to solve. The preservice teachers were responsible for solving the other members' problems. This gave each preservice teacher a variety of methods to consider when trying to anticipate students' thinking when writing their lesson plans.

To explain the process, I will describe the implementation of one of my favorite mathematical investigations: the Staircase problem. This and similar problems can be found in *Navigating through Algebra in Grades 6–8* (NCTM 2001). The lesson-planning process has proved to be a valuable learning experience for writing my own lesson plans as well as teaching methods students to write lesson plans. You the classroom teacher can benefit as well by choosing one of your favorite problems

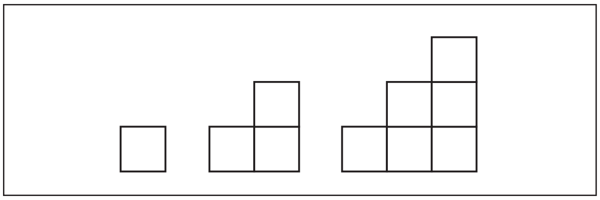

Fig. 1 Students are shown this diagram.

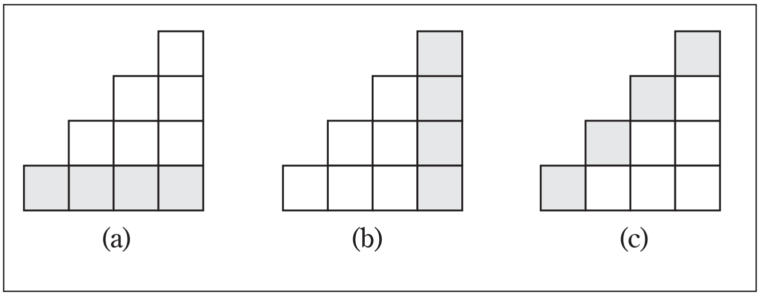

Fig. 2 Students add a horizontal, vertical, or a "staircase" row.

to solve. If you have already presented a problem to students, you will have a place to start, either writing or revising the lesson; you may already be able to anticipate some methods or problems your students might have.

Motivating Students to Solve the Problem

TO BEGIN INSTRUCTION, BAGS OF TWO-CENTIMETER cubes were distributed to groups of four or five students. The entire class is shown the first three arrangements of a growing pattern, illustrated in **figure 1**. Students are asked to picture and then describe what the 4th arrangement will look like without building or drawing it. Students differ in their descriptions of how four cubes are added to the third figure to produce the 4th. Some students add four cubes horizontally to form a fourth row **(fig. 2a)**; some add four cubes vertically to form a fourth column **(fig. 2b)**; and some add four cubes diagonally to form an additional staircase **(fig. 2c)**. I then ask them to *build* the 4th figure, directing their attention to the critical features of the pattern, such as the number of cubes in each arrangement and the construction from one arrangement to the next. **Table 1** contains the rest of the questioning sequence. I then present this problem for them to solve:

> How many cubes are in the 50th arrangement? We do not have enough cubes to build it, and adding the columns or counting the cubes one at a time would be too time-consuming. How else can we determine the total number of cubes in the 50th arrangement?

Presenting the problem does not end the discussion. Through experience and recommendations by the NCTM (2000), I include questions to help my students get started. Analyzing problems before attempting to solve them strengthens the chance that students

will persist in finding a solution, because "if they do not immediately know how to solve a problem, they will give up" (p. 259). Some students may know exactly what to do, whereas others have no idea. Once the problem is presented, I ask if anyone knows where to begin problem solving, not knowing if anyone will have an answer or what that answer might be. In this case, I plan for potential student responses as well as possible guidance for student thinking. **Table 2** lists some possible student responses along with guidance for the suggestions. The student suggestions are intended to give others, with no plan, a potential course of action to pursue. This critical element of problem solving allows all students the same opportunity to solve the problem without becoming frustrated or discouraged.

Student Problem Solving

AS STUDENTS BEGIN WORKING ON THE PROBLEM, it is important to step back and not interfere. I want them to struggle, but I also want them to be successful. Therefore, I must have a clear idea of where to begin questioning and where the questions are leading because the same feedback cannot be provided

TABLE 1
Possible Questioning Sequence to Stimulate the Solution Process

Questions	Responses and Follow-up Questions
Question 1: How many cubes are in the 4th arrangement?	*Student response 1:* 10. *Follow-up:* How did you determine it was 10 cubes? *Student response:* I added 4 + 3 + 2 + 1.
	Student response 2: 6. *Follow-up:* Can you tell me how you arrived at your answer? *Student response:* The second figure had 3 cubes, so twice that is 6. *Follow-up:* Can you confirm your answer by building **figure 2c**?
Question 2: Can you predict how many cubes the 5th figure will have?	*Student response:* 15. *Follow-up:* Did you use the same method you used for **figure 2c**? *Student response:* I knew **figure 2c** had 10 cubes and the next figure would have 5 more so I added 10 + 5 = 15. *Follow-up:* Can you build it?
Question 3: If this pattern is to continue, what do you think the 10th arrangement would look like? [I want the students to realize that the figure represents consecutive numbers.]	*Student response 1:* It would have 10 rows, with 10 on the bottom and then 9, 8, all the way to 1. *Follow-up:* Can someone else describe what he or she sees?
Question 4: Can someone describe the 20th arrangement? [Once the students are comfortable describing the growing pattern based on the arrangement, I am ready to introduce their problem for investigation.]	*Student response:* It would have 20 columns from heights 1 to 20. *Follow-up:* Can we build it? *Student response:* We don't have enough cubes to build it!
Question 5: Who thinks he or she could describe the 50th arrangement? [Students will likely recall that there is an algebraic formula, but it is unlikely that they will remember what it is. I want students to recognize that adding consecutive rows/columns or building the figure will be tedious and time-consuming.]	*Student response:* Its biggest row would be 50, and there would be 50 rows. The smallest row is 1. *Follow-up:* How could we figure out how many cubes are in the 50th figure? *Student response:* We know we don't have enough cubes to build it, so we could add the columns: 50 + 49 + 48 + · · · + 3 + 2 + 1. *Follow-up:* That worked for **figure 2c**, do you think it would also work for the 50th figure? If you don't have a calculator, are you going to add it by hand? *Student response:* Is there a formula we can use? *Follow-up:* What is the formula? Can we figure it out without a formula? Can we determine a formula based on our cube patterns?

TABLE 2
Helping Students Having Difficulties

Question	Responses and Follow-up Questions
Question: Does anyone have an idea of where to begin solving this problem? [Student 2 has a misconception.]	*Student response 1:* It looks kind of like a triangle. Maybe you can use that to figure it out. *Follow-up:* Do you know how to find the area of a triangle? *Student response 2:* The 5th figure had 15 cubes, so the 50th figure would have 10 times the number. *Follow-up:* Can we verify that using smaller figures such as from the 5th to the 10th? *Student response 3:* Maybe there is a pattern with the figure number and the number of cubes. *Follow-up:* Do you notice something from the first 5 figures we have?

TABLE 3
Probing Questions during Problem Solving

Questions	Reasoning
Explain to me how this group is solving the problem.	If you are not sure you or they know what they are doing
Will your method work for **figure 5**? Let's build it and see.	If a group has a wrong idea but does not realize it
Did I hear you say it reminds you of a triangle?	If a group wants to abandon an idea before completely investigating it
How did you figure out that the 10th figure had 55 cubes?	If you are trying to figure out what a group has done so you can begin helping from the point of problem
What have you tried that has worked or not worked? Let's look back at **figure 2c**. What is a way other than counting each block that you could determine the total number of blocks in the figure?	Students who still struggle with how to begin
Can one of your group members explain your method?	If it is unclear that all members are able to explain their solution
Can you demonstrate your equation on the 5th arrangement?	If a group knows a "formula" for determining the total number of blocks
Can everyone in your group explain the method you have used? Will this method work for any number of staircases? Can you write an equation based on your method that would work for any arrangement?	Students who solve the problem
How do you know your answer is right?	If you want students to prove their solution
Will there be a staircase that is made of exactly 100 blocks? Which arrangement in the sequence will be the first one to use more than 100 blocks to build?	If a group solves the problem, writes an equation, and every member can explain his or her thinking using concrete materials or representations

for all situations. Although a variety of actions are planned, I may not use some of the interventions. Some actions are for groups that may have misconceptions but do not realize it. Other actions are to help me figure out what the groups are thinking. While passively observing the groups and listening to their discussions, I must determine when it is appropriate to start asking questions, since my job is to provide help, not do the work for them.

Table 3 contains possible questions that are asked during this phase of the lesson, along with my reason for planning each particular question. For example, one group might add additional cubes to a given staircase arrangement to form a square (referring to the front face of the cubes). Although students might know how to determine the area of the square by multiplying the length of two sides, they might not know how to proceed. I begin questioning, based on what they have done so far, and must recognize if their struggles seem to get them nowhere. For example, if one group is sitting completely perplexed, I may ask if the staircase resembles a certain shape. Students often reply that the staircase resembles a triangle. This question will give them a place to start. For example, one group may reason that the 5th arrangement has 15 cubes, so the 50th arrangement must have 10 times that number, or 150. For this misconception, I would ask students to consider the 50th arrangement of blocks. How many cubes are in the last four columns? (They add 47 + 48 + 49 + 50, which equals 194.) Asking this question helps them address the reasonableness of their method.

Student Solutions

PRINCIPLES AND STANDARDS (2000) SUPPORTS PROBlem solving in which students develop and use a variety of strategies and approaches. Other Process Standards, such as Communication and Representation, also come into play. Students must communicate their ideas with other members of their group as well as to other groups in the class by representing their thinking.

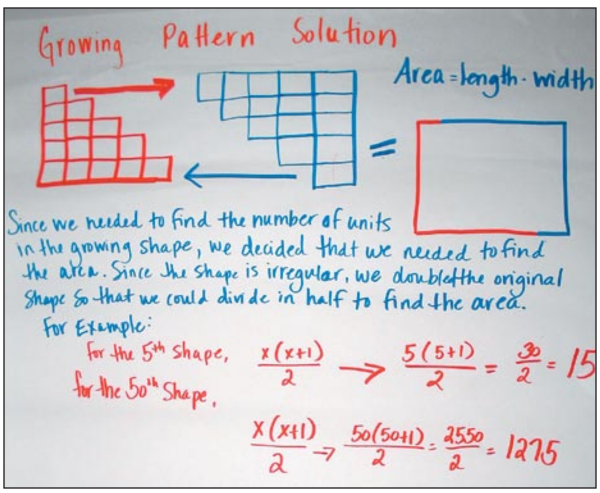

Fig. 3 A 5th arrangement is analyzed.

Each group is asked to present its methods and procedures on chart paper in a way that other students in the class can follow. **Figure 3** shows how one group used arrangement 5 to represent its thinking. They created an identical arrangement 5, rotated the new figure, translated it to their original arrangement, and produced a rectangle. Although the length stayed the same, the width increased by 1. They used the formula for determining the area of a rectangle ($l \times w$) to find that the front surface area (correlating to the number of cubes) of their rectangle is 30. They divided the area by 2, because they had combined two of the original arrangements, which resulted in the total number of cubes for one arrangement to be 15. The algebraic generalization they used to represent their thinking is

$$\frac{x(x+1)}{2},$$

where x represents the arrangement number, the number of squares in the longest row.

Figure 4 shows the "fill in the square" method: students did not increase the base or the height but added enough blocks to create a perfect square. They multiplied one side (known from the arrangement number) by the other side (which is the same, because they created a square) to determine the total

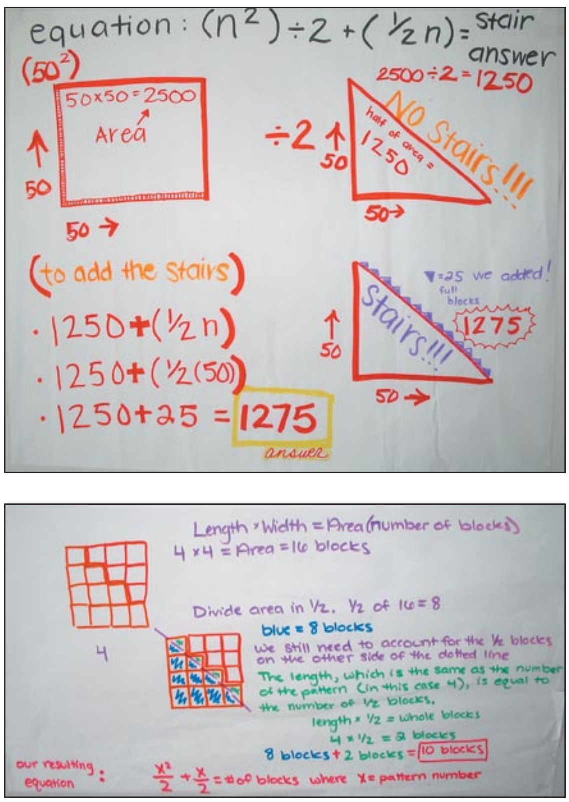

Fig. 4 The fill-in-the-square method is illustrated.

number of blocks in the square. They next divided the total number of blocks in their figure by 2, which gave them one-half the square. They found the number of cubes in their original figure but were missing half cubes along the diagonal. They added in the half cubes, or "steps," finding that the total number of pieces equaled one-half the base (or height). The algebraic generalizations resulting from these two groups are

$$\frac{n^2}{2}+\frac{1}{2}n \text{ and } \frac{x^2}{2}+\frac{x}{2},$$

where n and x represent the arrangement number.

Figure 5 represents "triangular" thinking; the group saw the arrangements as being triangles with tips leftover. A student placed an ink pen along the staircase diagonal and noticed that a right triangle was on one side of the pen and that tips, or one-half cubes, were on the other side. They knew the area of a triangle to be $(1/2)bh$, so they determined the area of the triangle first. Through experimentation, they determined the number of tips to equal one-half the base. Although the method may seem similar to the previous explanation, the thinking is different. The previous explanation recognized the arrangement as being one-half square plus half pieces of steps (along the diagonal), whereas this group saw a triangle plus tips (one-half the steps). One algebraic generalization representing this thinking is

$$\frac{1}{2}x^2+\frac{x}{2},$$

where x is the arrangement number.

Figure 6 uses the 4th arrangement to explain how many cubes are in an even-numbered arrangement. This group created a different figure using the same number of cubes by combining columns 1 and 3, making two columns of 4 with one column of 2 leftover. They discovered that by creating columns of the same height, the figure becomes half the length of the base, but the height stays the same, or

$$n\bullet\left(\frac{1}{2}n\right).$$

They always had 1/2 column (n) left, so they added $(1/2)n$ back in. The algebraic generalization representing their thinking is

$$n\bullet\left(\frac{1}{2}n\right)+\frac{1}{2}n,$$

where n represents the arrangement number.

The students did not extend the problem to odd arrangements. Although the next example also demonstrates even-numbered arrangements, the nature of the explanation shows how the number of cubes in an odd-numbered arrangement can be determined using similar thinking of combining columns. They represented their thinking using arrangement 6. Referring to **figure 7a,** they took away the largest column x of 6 cubes and left it alone. The remaining number of columns they now have is $x-1$, or $6-1=5$, columns. In **figure 7b**, they make as many columns of height x, or 6, as they can by adding column 1 of 1 cube to column 5 of 5 cubes and adding column 2 of 2 cubes to column 4 of 4 cubes. They are left with one column of 3, which they also make into a column of height 6 by splitting it in half and placing 1/2 on top of the other to make a column of height 6 and width 1/2 **(fig. 7c)**. This maneuvering of cubes resulted in a rectangular array of height 6 and width 2.5. The

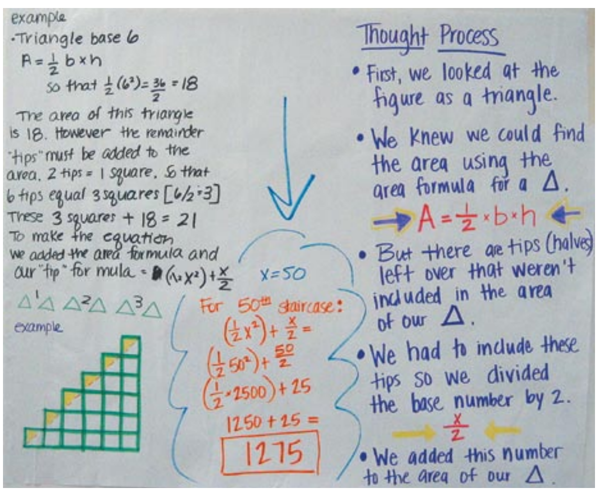

Fig. 5 Leftover triangular "tips" are part of the calculation.

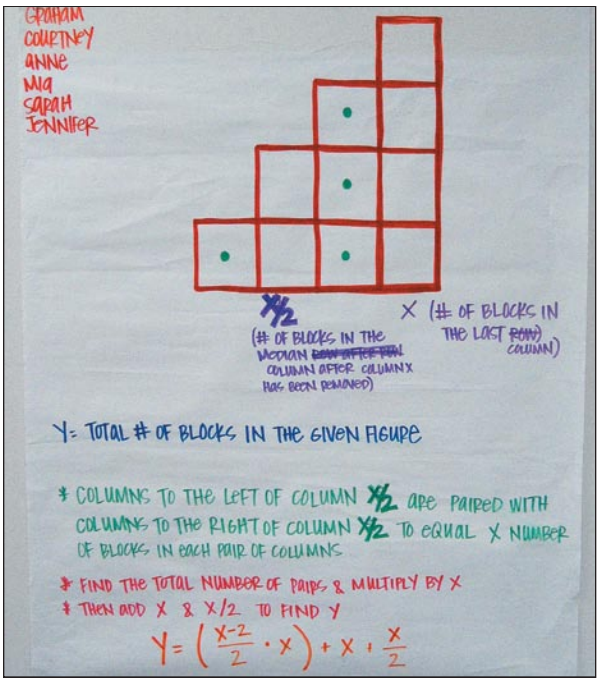

Fig. 6 Students added columnar amounts algebraically.

students realized that they now had $(x-1)/2$ columns of x. In **figure 7d**, they added the original column of x back into their representation. The algebraic generalization they used to represent their thinking is

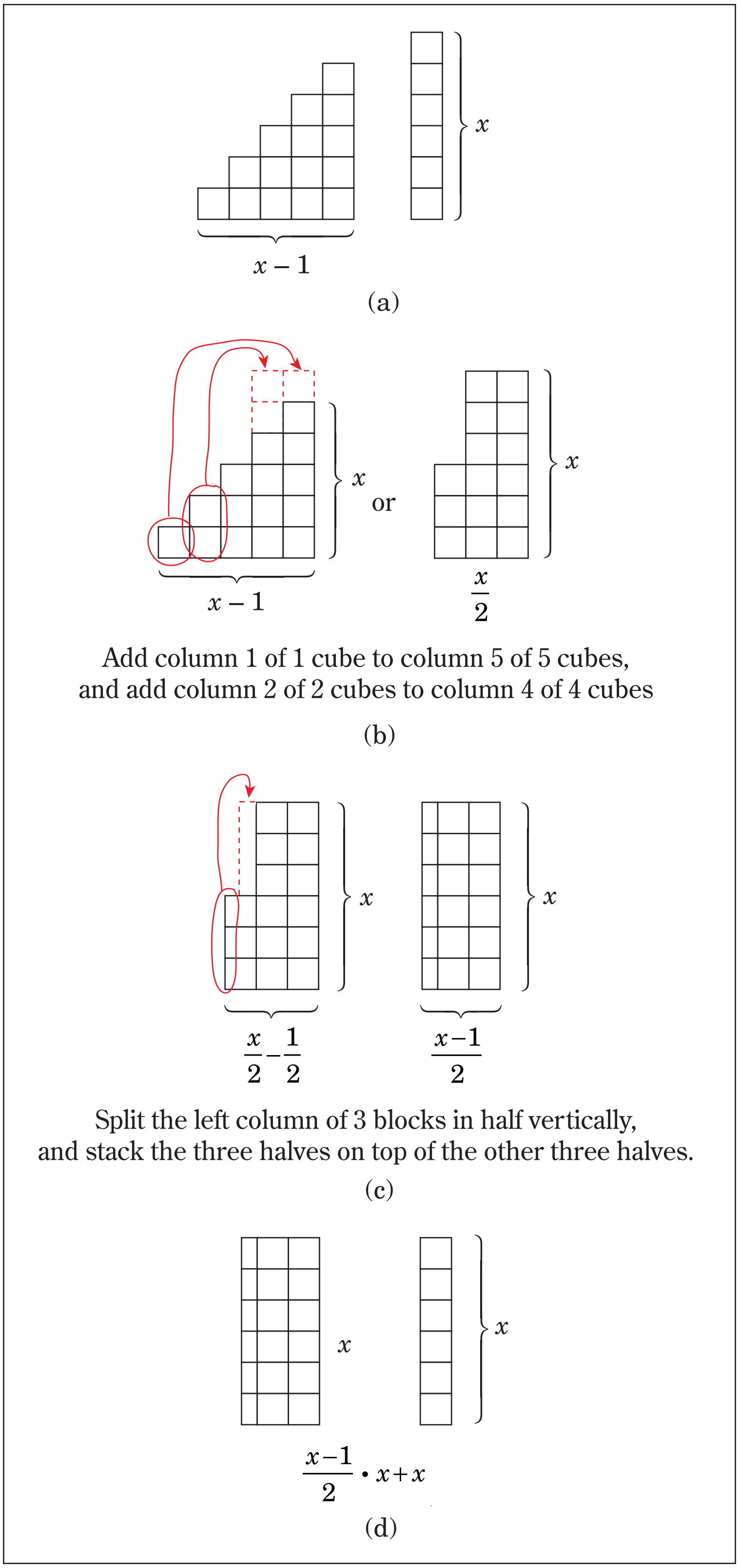

Fig. 7 Illustrations and algebra comprise this solution method.

$$\left(\frac{x-1}{2}\cdot x\right)+x.$$

Some alternate generalizations are

$$\frac{1}{2}bh+\frac{1}{2}b \text{ and } x\left(\frac{x}{2}\right)+\frac{x}{2},$$

in which b, h, and x all refer to the arrangement number.

The variety of generalizations produced by the students is quite impressive. The generalizations yield the same answers for the 50th (or nth) arrangement, yet they look different because they represent how each group was thinking about the problem. I am always excited to see the different generalizations and variables used to solve the problem. Students often wonder how different generalizations, such as

$$\frac{1}{2}bh+\frac{1}{2}b \text{ and } x\left(\frac{x}{2}\right)+\frac{x}{2},$$

can represent the same problem. We investigate this matter during the subsequent lesson as we address the meaning of variable in all the generalizations and how the generalizations can be simplified.

Lessons Learned

I LEARNED THESE CRITICAL LESSONS THROUGHout this experience.

- Prior questioning—The more I questioned *prior to* giving the problem, the less help the students needed from me *during* problem solving.
- Anticipating student responses—I do not have to make on-the-fly decisions because I have already anticipated problems, actions, and conflicts as well as planned how to respond. Without anticipating every situation, I must be prepared for many possibilities.
- Signaling errors—Warnings to the students for potential errors are ineffective if they are not accompanied by a sequence of possible actions.
- Student engagement—When students solve interesting problems with multiple solutions, the class is eager to hear the results; they recognize that other methods may be far easier or much more complicated than their own.
- Student presentations—It is not necessary for every group to share its results, especially when the methods are the same. This speeds up the sharing process and eliminates repetitive solutions.
- Mathematical accuracy—When students share their solutions and methods, I am responsible for verifying the correct mathematics and ensuring

that the students' reasoning matches the equations and explanations.

- Using formulas—Students who can provide a formula to solve a problem are usually unable to explain why it works, either using the cubes or other representations.
- Significance of problem solving—Students learned the importance of problem solving when they cannot recall a formula to solve the problem. They gained confidence through their struggles and felt a sense of accomplishment when they solved the problem.

Conclusion

A CLEAR EMPHASIS ON STUDENTS' THINKING IS critical to lesson planning. In this experience, I identified a problem, but the lesson revolved around the ideas that the students generated through their own investigations. The problem could be approached in more than one way, and I was prepared for different approaches (both correct and incorrect), which enabled me to orchestrate the work of the students without taking over the process of thinking for them.

I do not write detailed plans of this extent for all of my lessons, but when a particular problem or task lends itself to more than one strategy or method, this form of planning helps me prepare for crucial elements. Before learning to plan for student thinking, all the methods and strategies my students shared were the same because they were mirroring my solution strategy. By carefully choosing problems, anticipating students' thinking and questions, and preparing a course of action, I am able to use problem solving to further my mathematical goals for the class.

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